# Tutorial 7

written by Zhiwen Zhang

9th week

1. Find a potential function $f$ for the field $\vec{F}$

$$\vec{F} = 2x\vec{i} + 3y\vec{j} + 4z\vec{k}$$

Suppose $\vec{F} = \nabla f$

$$\frac{\partial f}{\partial x} = 2x \qquad \frac{\partial f}{\partial y} = 3y \qquad \frac{\partial f}{\partial z} = 4z$$

$$\frac{\partial f}{\partial x} = 2x \Rightarrow f = x^2 + C(y,z)$$

$$\frac{\partial f}{\partial y} = 3y \Rightarrow f = \frac{3}{2}y^2 + C(x,z)$$

$$\frac{\partial f}{\partial z} = 4z \Rightarrow f = 2z^2 + C(x,y).$$

Put them together,

$$f = x^2 + \frac{3}{2}y^2 + 2z^2 + C$$

2. Evaluate the the integral

$$\int_{(1,1,2)}^{(3,5,0)} yz\,dx + xz\,dy + xy\,dz$$

Let $\vec{F}(x,y,z) = yz\,\vec{i} + xz\,\vec{j} + xy\,\vec{k}$

$\frac{\partial(yz)}{\partial y} = \frac{\partial(xz)}{\partial x} = z \qquad \frac{\partial(xz)}{\partial z} = \frac{\partial(xy)}{\partial y} = x \qquad \frac{\partial(xy)}{\partial x} = \frac{\partial(yz)}{\partial z} = y.$

So $\vec{F}$ is a conservative vector field.

Suppose $\vec{F} = \nabla f$.

We have $f(x,y,z) = xyz + C$

Then

$$\int_{(1,1,2)}^{(3,5,0)} yz\,dx + xz\,dy + xy\,dz$$

$$= f(3,5,0) - f(1,1,2)$$

$$= -2$$

3. Show that this integral does not depend on the path taken from $A$ to $B$.

$$\int_A^B \frac{x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z}{\sqrt{x^2+y^2+z^2}}$$

Let $M(x,y,z) = \dfrac{x}{\sqrt{x^2+y^2+z^2}}$ $\quad N(x,y,z) = \dfrac{y}{\sqrt{x^2+y^2+z^2}}$ $\quad P(x,y,z) = \dfrac{z}{\sqrt{x^2+y^2+z^2}}$

then

$$\frac{\partial M}{\partial y} = \frac{-xy}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \frac{\partial N}{\partial x}$$

$$\frac{\partial M}{\partial z} = \frac{-xz}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \frac{\partial P}{\partial x}$$

$$\frac{\partial N}{\partial z} = \frac{-yz}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \frac{\partial P}{\partial y}$$

Hence

$\dfrac{xdx + ydz + zdz}{\sqrt{x^2+y^2+z^2}}$ is an exact differential form

4. Find a potential function for $\vec{F}$.

$$\vec{F} = (e^x \ln y)\vec{i} + \left(\frac{e^x}{y} + \sin z\right)\vec{j} + (y \cos z)\vec{k}$$

Suppose $\vec{F} = \nabla f$

$$\frac{\partial f}{\partial x} = e^x \ln y \Rightarrow f = e^x \ln y + C(y,z)$$

$$\frac{\partial f}{\partial y} = \frac{e^x}{y} + \sin z \Rightarrow f = e^x \ln y + y \sin z + C(x,z)$$

$$\frac{\partial f}{\partial z} = y \cos z \Rightarrow f = y \sin z + C(x,y)$$

Put them together

$$f(x,y,z) = e^x \ln y + y \sin z + C$$

5. Suppose that $\vec{F} = \nabla f$ is a conservative vector field and

$$g(x, y, z) = \int_{(0,0,0)}^{(x,y,z)} \vec{F} \cdot \mathrm{d}\vec{r}$$

Show that $\nabla g = \vec{F}$.

$$\nabla g = \nabla\left( f(x,y,z) - f(0,0,0)\right)$$

$$= \nabla f$$

$$= \vec{F}$$